Ito integral computation

Math 485

December 6, 2013

1 Goal:

To compute the explicit formula for some Ito integrals and verify the fact that Ito integral is a martingale with respect to the filtration generated by Brownian motion for "nice" integrands.

2 $\int_0^t B_s dB_s$:

Since intuitively, the "antiderivative" of B_t is B_t^2, we apply Ito's formula to B_t^2:

$$dB_t^2 = 2B_t dB_t + dt.$$

Hence

$$B_t^2 - B_0^2 = 2\int_0^t B_s dB_s + \int_0^t ds,$$

which implies

$$\int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t).$$

This finishes the computation of $\int_0^t B_s dB_s$. To verify that it is a martingale we need to verify for $s \le t$

$$E(\int_0^t B_r dB_r | B_s) = \int_0^s B_r dB_r,$$

which is equivalent to verifying

$$E(B_t^2 - t | B_s) = B_s^2 - s. \tag{1}$$

Thus we need to compute $E(B_t^2|B_s)$. We have

$$\begin{aligned} E(B_t^2|B_s) &= E([B_s + (B_t - B_s)]^2|B_s) \\ &= E(B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2|B_s) \\ &= B_s^2 + (t - s), \end{aligned}$$

(By remembering that $B_t - B_s$ is independent of B_s).

It is easy to verify that (1) is true now.

3 $\int_0^t e^{B_s} dB_s$:

From the technique of section 2, it is easy to see that we can compute $\int_0^t B_s^k B_s$ and verify that it is a martingale for any integer k. In this section we'll compute a more challenging integral, namely $\int_0^t e^{B_s} dB_s$.

Again the idea is to apply Ito's formula to the "antiderivative" of e^{B_t}, which is e^{B_t}.

$$de^{B_t} = e^{B_t} dB_t + \frac{1}{2} e^{B_t} dt.$$

Thus

$$e^{B_t} - e^{B_0} = \int_0^t e^{B_s} dB_s + \frac{1}{2} \int_0^t e^{B_s} ds.$$

So

$$\int_0^t e^{B_s} dB_s = e^{B_t} - 1 - \frac{1}{2} \int_0^t e^{B_s} ds.$$

This finishes the computation of the integral $\int_0^t e^{B_s} dB_s$. Notice then by computation, we mean rewrite the Ito integral into other expressions that do not invole the integral with respect to dB_t. To verify that $\int_0^t e^{B_s} dB_s$ is a martingale, we need to verify for $s \leq t$

$$E\big(e^{B_t} - 1 - \frac{1}{2} \int_0^t e^{B_r} dr|B_s\big) = e^{B_s} - 1 - \frac{1}{2} \int_0^s e^{B_r} dr. \tag{2}$$

So we need to compute two things: $E(e^{B_t}|B_s)$ and $E(\int_0^t e^{B_r} dr|B_s)$. The first computation is standard:

$$\begin{aligned} E(e^{B_t}|B_s) &= E(e^{B_s} e^{B_t - B_s}|B_s) \\ &= e^{B_s} e^{\frac{1}{2}(t-s)}. \end{aligned}$$

The second computation requires more attention:

$$E(\int_0^t e^{B_r} dr | B_s) = \int_0^t E(e^{B_r} | B_s) dr$$

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Now for $r \leq s, E(e^{B_r}|B_s) = e^{B_r}$. But for $r \leq s, E(e^{B_r}|B_s) = e^{B_s} e^{\frac{1}{2}(r-s)}$ (by exactly the same computation just above). Thus

$$\begin{aligned} E(\int_0^t e^{B_r} dr | B_s) &= \int_0^s e^{B_r} dr + \int_s^t e^{B_s} e^{\frac{1}{2}(r-s)} dr. \\ &= \int_0^s e^{B_r} dr + e^{B_s} e^{-\frac{s}{2}} \left[2e^{\frac{1}{2}r}\right]\Big|_s^t \\ &= \int_0^s e^{B_r} dr + 2e^{B_s} \left[e^{\frac{1}{2}(t-s)} - 1\right]. \end{aligned}$$

It is easy to verify the equality (2) now (See Homework 7 Problem 1 b).